Important Concepts and Definition

Without definitions, there are nothing to do. Please at least recite definitions.

Definition 1 (Upper Bound/Lower Bound)**.** Let X be a non-empty subset of $\mathbb{R}$.

- $u \in \mathbb{R}$ is said to be an *upper bound* of X if $u \geq u$ for all $x \in X$.
- $l \in \mathbb{R}$ is said to be a *lower bound* of X if $l \leq x$ for all $x \in X$.
- X is said to be *bounded above* if X has an upper bound.
- X is said to be *bounded below* if X has a lower bound.
- X is said to be *bounded* if X is bounded above and bounded below.
- X is said to be *unbounded* if X is not bounded.

Definition 2 (Supremum/Infimum)**.** Let X be a non-empty subset of $\mathbb{R}$.

- The least upper bound of X, denoted by $\sup X$, is called the *supremum* of X.
- The greatest lower bound of X, denoted by $\inf X$, is called the *infimum* of X.

Definition 3 (Sequence)**.** A *sequence* of real numbers X is a function $X : \mathbb{N} \to \mathbb{R}$. A sequence is usually denoted by

$$(x_n) \quad \text{or} \quad (x_1, x_2, x_3, ...),$$

where $x_n = X(n)$ denotes the *n-th term* of the sequence for each $n \in \mathbb{N}$.

Definition 4 (Convergent Sequence)**.** Let (x_n) be a sequence of real numbers.

- $x \in \mathbb{R}$ is said to be a *limit* of (x_n), or (x_n) is said to *converge* to x, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

  $$|x_n - x| < \varepsilon, \quad \forall n \geq N.$$

  In this case, denote $x = \lim x_n$.
- (x_n) is said to be *convergent* if it has a limit.
- (x_n) is said to be *divergent* if it is not convergent.

Definition 5 (Bounded Sequence)**.** Let (x_n) be a sequence of real numbers.

- (x_n) is said to be *bounded* if there exists a real number $M > 0$ such that

  $$|x_n| \leq M, \quad \forall n \in \mathbb{N}.$$

- (x_n) is said to be *unbounded* if (x_n) is not bounded.

Definition 6 (Monotone Sequence)**.** Let (x_n) be a sequence of real numbers.

- (x_n) is said to be *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
- (x_n) is said to be *decreasing* if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.
- (x_n) is said to be *monotone* if (x_n) is increasing or decreasing.

Definition 7 (Subsequence)**.** Let (x_n) be a sequence of real numbers and (n_k) is a strictly increasing sequence of natural numbers. The sequence (x_{n_k}) is called a *subsequence* of (x_n).

Definition 8 (Cauchy Sequence)**.** A sequence of real numbers (x_n) is said to be *Cauchy* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon, \quad \forall n, m \geq N.$$

Important/Useful Theorems and Propositions

The following are the main theorem and useful theorems in this course. Please remember their statements well. Minor differences may cause inaccuracy in your proofs. You should also understand their proofs, and even be able to prove them.

Proposition 1. *Let X be a non-empty subset of $\mathbb{R}$ and $u \in \mathbb{R}$ be an upper bound of X. The following are equivalent:*

- *$u \in \mathbb{R}$ is the supremum of X.*
- *If $v < u$, then v is not an upper bound of X.*
- *If $v < u$, then there exists some $x \in X$ such that $x > v$.*
- *For any $\varepsilon > 0$, there exists some $x \in X$ such that $x > u - \varepsilon$.*

The Completeness Property of $\mathbb{R}$. *Every non-empty subset of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Archimedean Property. *The following are the most used versions:*

- *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.*
- *For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $1/n_\varepsilon < \varepsilon$.*

Nested Interval Property. *If $I_n = [a_n, b_n]$ is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. Moreover, if the lengths $b_n - a_n$ of I_n satisfies*

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

then the number $\xi \in \mathbb{R}$ is unique.

Theorem 1. *A convergent sequence is bounded.*

Squeeze Theorem. *Let (x_n), (y_n) and (z_n) be sequences of real numbers. Suppose that*

$$x_n \leq y_n \leq z_n, \quad \forall n \in \mathbb{N},$$

and $\lim x_n = \lim z_n$. Then (y_n) is convergent and

$$\lim x_n = \lim y_n = \lim z_n.$$

Monotone Convergence Theorem. *A monotone sequence of real numbers (x_n) is convergent if and only if it is bounded. Furthermore,*

- *If (x_n) is bounded and increasing, then $\lim x_n = \sup\{x_n : x \in \mathbb{N}\}$.*
- *If (x_n) is bounded and decreasing, then $\lim x_n = \inf\{x_n : x \in \mathbb{N}\}$.*

Theorem 2. *Let (x_n) be a sequence of real numbers. If (x_n) is convergent, then any subsequence of (x_n) converges to the same limit as (x_n).*

Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence.*

Cauchy Convergence Criterion. *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Divergence Criteria. *If one of the following holds, then the sequence of real numbers (x_n) is divergent:*

- *(x_n) is unbounded.*
- *(x_n) has two subsequences with unequal limits.*
- *(x_n) is not Cauchy.*